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Oblique wave incidence on a plane beach: the classical problem revisited

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The non-hydrostatic description of three-dimensional waves incident over a plane beach at a long straight coastline is considered in terms of the inverse Kontorovich–Lebedev integral transform. This is seen as a natural extension to earlier work by the author where the two-dimensional (normal incidence) flow is expressed as an inverse Mellin transform, and similar simplifications in the description here are encountered. In particular computations are undertaken for a variety of beach slopes of the form $\alpha = \pi/2m$ where m is an integer and for a range of incidence angles. These computations have previously only been practical for beaches whose slope α is regarded as asymptotically small thereby allowing versions of the mild-slope equation to be used. For the chosen slope angles, the solution is established with rigour and methods of estimating near- and far-field asymptotics arise naturally in this discussion. For the case of perfect reflection, a previously known solution is recovered in closed form as a finite sum of exponential terms, and a shoreline 'amplification factor' a_γ is considered for these waves and is computed for a range of beach slopes through the entire spectrum of incidence angles. It is shown analytically that, in the limit of normal incidence, the value of a_γ approaches the well-known classical result $a_0 = m^{1/2}$ and, for glancing incidence, Whitham's (1979) result is confirmed where the value approaches either 1 or 0 depending on whether the beach angle is or is not an angle at which a new Ursell edge wave mode appears (m odd).

As applications of the new development, comprehensive near-field expansions for arbitrary reflection are written and verified by computation. These permit the construction of refracted wavefronts and wave rays for arbitrary beach slope without the usual phase velocity assumptions. Instability is indicated at very oblique incidence where nonlinear modelling (Peregrine & Ryrie 1983) predicts 'anomalous refraction'. Results are presented graphically and computation of derivatives of the potential enables estimation of the (second order) *set-down* seaward of the breaker zone. This is found to decrease as wave attack becomes increasingly oblique.

1. Introduction

The classical problem of waves attacking obliquely a long straight coastline where the beach is plane and of arbitrary declination α has been previously described in great detail by many authors. The justification for revisiting the problem rests largely with the comparative computational 'inaccessibility' of the solutions. Hanson (1926), Peters (1952) and Roseau (1952) are some of the early works on small-amplitude waves on a perfect fluid in irrotational motion (Hanson's work was restricted to the bounded standing wave). These works are specific to the 'continuous spectrum' case (oscillatory structure in the off-shore direction) whilst the 'discrete spectrum case', where waves propagate along the shore, was first fully investigated by Ursell (1952) and later by Roseau (1958). This case leads to the well-known edge waves which have been subsequently studied by many other authors, e.g. Minzoni & Whitham (1977) who discussed the nonlinear excitation of edge waves by continuous spectrum standing waves, Evans (1988, 1989) who, using a linear framework, discussed first the generation of waves by bottom protrusions or surface pressure distributions and later the general bounded edge wave solutions under a generalized (Robin-type) bed condition. More recently Miles (1990 *a*) and Blondeaux & Vittori (1995) also discussed the nonlinear excitation of edge waves. A fuller discussion using techniques of the present paper will, for the case of the discrete spectrum, be deferred to a later work, but readers will find an excellent up-to-date survey on edge waves given by Evans & Kuznetsov (1997) (see also Kuznetsov *et al.* 1998).

A succinct overview of Peters' work is given by Stoker (1958) whilst a more recent (but perhaps lesser known) work by Whitham (1979, Chap. 7) describes a similar approach but contains a number of interesting additional ramifications; the reader can refer to any of these works for the basic formulation which will be adopted here in § 2. Whilst the derivation of the solution provided by Peters and (for the special case of the slope angles considered herein) Whitham is one of considerable ingenuity, the outcome is an extremely complicated form of solution and one which does not lend itself to computation with any degree of simplicity except in the case of the regular standing wave (see below). Indeed, it is noteworthy that whilst the classical two-dimensional (normal incidence) solutions (Stoker 1958) are often cited and used quantitatively by authors (e.g. Keller 1961; Blondeaux & Vittori 1995) there appears to be little use made of the more general oblique incidence solutions discussed first by Peters and Roseau and later by Lauwerier (1959). Paradoxically, the latter work throws some light on this issue. On the one hand, Lauwerier is unable properly to reconcile the new solution with that of Peters, but for the special case of the vertical cliff at least the well-known solution of Weinstein (1949) is recovered. Lauwerier also remarks that the new solution is obtained in a form which makes it perhaps amenable to further treatment. The test of time has evidently not substantiated this prophecy but this may be as much due to progress in computational models in the last few decades as to any particular deficiency with Lauwerier's solution. Those computational models, however, need validating and (numerical) calibrating (see for example Ehrenmark & Williams 1996 on the two-dimensional model) and it is here that the classical analytical models may retain a fundamental importance. The further quest for a solution which may be regarded as more amenable to treatment than any of the previously known solutions therefore remains thoroughly justified and one of the main objectives of the present paper is to establish, with rigour, a new description of the solution and to follow this up with a demonstration of its computational applicability by a series of calculations both from the full solutions and from asymptotic estimates of it. In particular, a comprehensive near-field expansion, which has not previously been written, is derived, computed and compared with the full solution. In this way (with coding available from the author on request) both computational modellers and those seeking more efficient mathematical approximations (e.g. parabolic models) are provided with a further validation test for three-dimensional wave propagation. This provision is in the form of a simple inverse integral transform or a uniformly convergent series expansion which represent the main results of the present work.

Whitham (1979, Chap. 7) exposes some of the restrictions of the classical models (e.g. Peters 1952; Ursell 1952) although, curiously, no mention is made of the contributions of Roseau (1952, 1958). In his work, Whitham shows that both edge waves and oblique waves are obtainable from the same model. This, along with many other derivations, is consistent with the results of the present model. Symptomatic of the various models, however, is the different degrees of difficulty in achieving specific outcomes. Most noteworthy of these, perhaps, is the difficulty of establishing e.g. Weinstein's solution (see § 6) from Peters' and Whitham's model compared to the present model. On the other hand, the relationship, for example, between near- and far-field asymptotic values seems to be more easily deduced by Whitham. These aspects are discussed more fully in the text.

In earlier work e.g. Ehrenmark (1989) the author showed that, in the case of normal incidence, the classical descriptions, e.g., Isaacsson (1950), Weinstein (1949), Stoker (1947), were considerably simplified by deployment of the inverse Mellin transform as an alternative solver. Indeed, more recently (Ehrenmark 1994, 1996) the author has shown also that a second-order analysis is possible under this description and this has been used to compute both set-down and second-order currents so driven, albeit under the restriction of validity to beyond some small distance from the shoreline, owing to assumptions of linearization and non-breaking waves. Such a theory (non-breaking) can be expected to be more relevant for steeper beaches where the reflection is probably considerable and where the conventional shallow-water theory becomes largely inapplicable and is replaced by a non-hydrostatic theory. In order not to compromise the mathematical generality of the solutions generated, the merits of the logarithmically (shoreline) singular solution will not be discussed here, except to record that whilst exclusion would render the solution bounded there, it would also imply perfect reflection, which could only be anticipated a good approximation on a beach of extreme steepness. An alternative viewpoint is to allow partial, or zero, reflection and then recognize that, the model being frictionless, logarithmically large shoreline values would in reality be damped out by friction and turbulent losses following breaking. In this way, the theory could be expected to be adequate at least seaward of the breaker zone or, for non-breaking waves, seaward of the dissipation zone (Miles 1990*b*; Ehrenmark 1996). Thus, along with the earlier treatments of this problem, both the 'regular' and 'singular' wave components will be studied so that arbitrary solutions can be constructed as required.

The purpose of the present work therefore, is to examine the oblique incidence problem, similar to the two-dimensional treatment (Ehrenmark 1989), but in this case handled by application of the inverse Kontorovich–Lebedev transform (K–L hereafter). The general formulation of the problem is restated in § 2 and the K–L ansatz is developed in § 3. This leads to a second-order functional equation whose solutions, for the special slope angles α ($= \pi/2q$, $q \in \boldsymbol{N}$), are constructed in § 4. The rigorous development of the initial solution is found to have shortcomings unless the far-field behaviour is first subtracted out and this technique is used to establish a proof of the solution (Appendix D). In § 5 the particular case of the perfectly reflected wave is examined and it is shown, with some details deferred to an Appendix, that the solution remains obtainable in closed form for arbitrary angle of incidence. This result had been previously observed (see Roseau 1952) but surprisingly is not discussed by either Peters or Stoker. Moreover, Bruce (1998) has shown that, starting from Peters' (1952) solution, considerable labour is required to obtain the full details of this result even for the simplest case of a beach of 45° slope. Roseau (1951, 1952) did write this solution but obtained it in a fashion similar to that used by Weinstein (1949) for the vertical cliff.

Sections 6 and 7 examine respectively the cases of progressing waves against a vertical cliff and on a beach. In the former case, it is shown that the special solution derived by Weinstein (1949) is easily recovered and in the latter case the solution is written in a form which, for vanishingly small incidence angles, is seen to reduce precisely to the potential functions written by Stoker (1958, p. 84) for the two-dimensional problem. It is believed that this form is preferable to some earlier forms (e.g. Peters 1952 or Roseau 1952) and comprehensive calculations are undertaken to demonstrate this. An expression is also written for amplification at the shoreline of waves having essentially unit amplitude at infinity (being standing waves, the notion of amplitude at infinity has to be taken to mean the maximum possible amplitude). Computations are undertaken for varying angles of incidence γ and excellent agreement is noted with the classical two-dimensional normal incidence solution when γ is taken to be very small. It is established in Appendix C that the expression for the amplification factor reduces, in this limit, to the classical form noted by e.g. Friedrichs (1948), Keller (1961). This result is considered of special interest as are also the extended calculations for glancing incidence which confirm that, as $\gamma \to \frac{1}{2}\pi$, the shoreline 'amplitude' approaches zero except for the special slope angles $\frac{1}{6}\pi$, $\frac{1}{10}\pi$, $\frac{1}{14}\pi, \ldots$ which are precisely those angles at which new discrete modes of the Ursell (1952) 'edge waves' appear when the frequency is regarded as fixed. For these angles, the value unity is instead obtained for the shoreline amplitude. This observation was first made by Whitham (1979) but without any specific computations. Moreover, Whitham, whilst explaining the phenomenon in terms of phase shifts at the shoreline causing alternately reinforcement and cancellation of wave components there, remained puzzled by the strange behaviour, remarking on the fact that one would not expect rapid changes in (potential) as the bed angle is gently decreased at already small values. The present work does not throw any new light on this issue, except to note that, at these angles, the waves are identical to the cut-off modes that form part of Ursell's edge wave description and that these modes will therefore have the same amplitude at the shoreline as they do at infinity; an observation which, evidently, has only therefore been indirectly observed previously. It is also remarked that there may be a need to compute with accuracy the original solutions presented by Peters which are valid for a continuous spectrum of bed-slope angles. In this way, it may be possible to determine more accurately the manifestation in general of the 'odd' behaviour. The prospect of doing this is a little daunting but the present solution and related computations (having been thoroughly verified) will provide the necessary validation tool for whatever numerical procedure is invoked in the more general case. In connection with these observations as $\gamma \to \frac{1}{2}\pi$, it may be noted from Peregrine & Ryrie (1983) (see also Peregrine 1983 and Ryrie & Peregrine 1982) that the linear refraction theory appears to have a zero amplitude range of applicability at 90° incidence. As discovered by Peregrine & Ryrie, using a nonlinear finite-amplitude theory, anomalous refraction appears at glancing incidence and wave rays will, in some circumstances, refract back toward more shore-parallel incidence. The linear theory cannot of course account for the anomalous refraction but it may be worth remarking that, for values of γ in excess of about 70°, there appears (with non-hydrostatic linear theory having replaced the more usual classical linear Airy theory) a certain amount of instability in the calculated turning of the wave ray. Some evidence of this can be seen in figure 6(a, b). The Section is concluded by summarizing one of the main results of the paper, namely the full expression for the potential of an incoming progressing wave.

Near-field expansions are obtained by residues providing a method of approximating readily (in particular) the singular solution near the shoreline. These expansions,

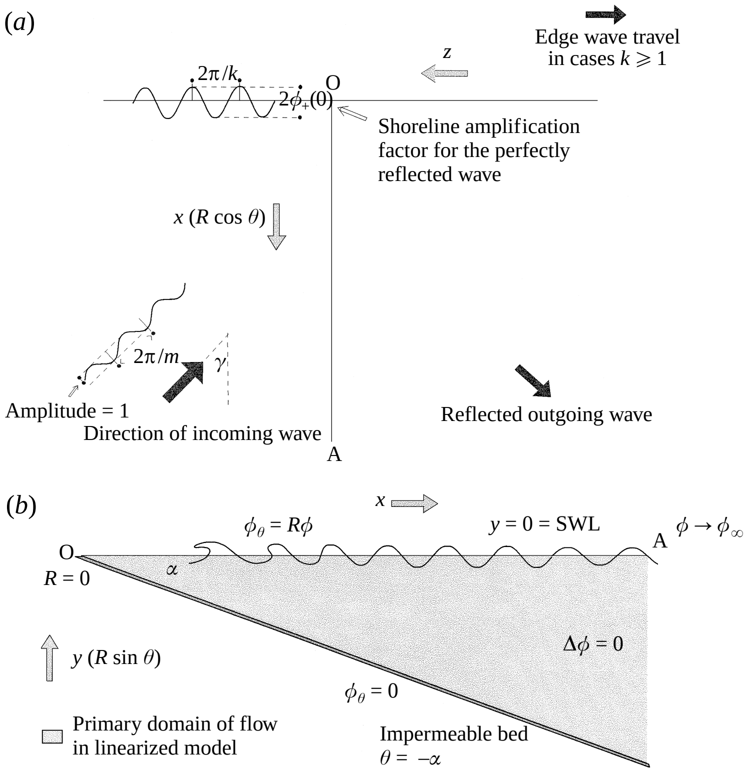

FIGURE 1. Geometry of the problem: (*a*) Plan view, (*b*) section view.

which have not previously been established, are given in § 8 and are augmented by a graph to indicate the range of validity that can be expected depending on the number of terms taken. Agreement with earlier results for shore normal incidence is demonstrated by taking an extremely small incidence angle. The results are used to compute a refraction diagram for 45° incidence on a beach of slope 6°. Finally, in § 9, the application to an expression for the wave-induced set-down seaward of the breaker zone is considered (Longuet-Higgins & Stewart 1963). This is computed for a 30° beach slope for a range of incidence angles. Some concluding remarks are made in § 10.

2. Formulation

The geometry is defined by the z-axis being taken along the shoreline, x directed out to sea and y-taken vertically upwards with $y = 0$ as the (still water level) SWL. A schematic diagram is given as figure 1. Monochromatic waves of angular frequency ω and potential 'amplitude' a are then described in deep water by the potential

$$\Phi_\infty = a\,\mathrm{Re}\{\phi_\infty \exp(\mathrm{i}\omega t)\},$$

where

$$\phi_\infty = \exp\{\mathrm{i}(nx + kz)\}\mathrm{e}^{my}$$

and the angle of departure γ from normal incidence is given by $\cos\gamma = n/m$, $\sin\gamma = k/m$. To satisfy the Laplace equation it is necessary to take

$$n^2 + k^2 = m^2. \tag{2.1}$$

Peters (1952) looks for harmonic functions which have this behaviour at infinity and which in addition satisfy the normal kinematic and dynamic conditions on the surface, a condition of no normal flow on the bed together with a suitable boundedness condition at the shoreline. It is unnecessary to dwell on the origin of these conditions; all are well documented by Stoker (1958) and the system is written explicitly but using instead cylindrical polar coordinates with $\theta = 0$ as the SWL and $r = 0$ as the shoreline. Thus the shore normal coordinate is $r \cos\theta$. It is convenient to use non-dimensional variables based on the angular frequency ω and wavelength at infinity $2\pi/m$, $m = \omega^2/g$. The new (2DHV) radial coordinate is $R = r\omega^2/g$. The dimensional variables are also non-dimensionalized using off-shore wavenumber and phase speed. Writing $\Phi = a\,\mathrm{Re}\{\phi(R,\theta)\exp(\mathrm{i}t)\mathrm{e}^{\mathrm{i}kz}\}$, where $g^2\Phi/\omega^3$ is the dimensioned potential, the system for the dimensionless potential function ϕ may be taken in the form

$$(\varDelta - \kappa^2)\phi = 0; \{0 < R < \infty, -\alpha < \theta < 0\}, \tag{2.2}$$

$$\phi_\theta(R, -\alpha) = 0; \{0 < R < \infty\}, \tag{2.3}$$

$$\phi_\theta(R, 0) = R\phi(R, 0); \{0 < R < \infty\}, \tag{2.4}$$

$$\lim_{R\to\infty}\{\phi - \phi_\infty\} = 0; \{-\alpha < \theta < 0\}, \tag{2.5}$$

$$\lim_{R\to 0}\{\phi/\ln R\} = -\lambda; \{\lambda\,\text{constant}\}. \tag{2.6}$$

In the above $\kappa = k/m$, the suffix θ denotes a partial derivative and it is understood that the Laplacian is also dimensionless.

3. Construction of a solution

A formal solution of (2.2) which satisfies (2.3) is

$$\phi(R,\theta) = \lambda K_0(\kappa R) + \int_0^\infty A(s)\cosh s(\theta+\alpha) K_{is}(\kappa R)\,\mathrm{d}s, \tag{3.1}$$

where K_ν is Macdonald's function (Watson 1944) and λ is a suitable constant which will later be chosen so that condition (2.4) is satisfied. Then using the inversion formula for K–L (Oberhettinger 1972, p. 241) it follows, again purely formally, that

$$A(s)\cosh s(\theta+\alpha) = \frac{2s}{\pi^2}\sinh \pi s \int_0^\infty \frac{\{\phi(R,\theta) - \lambda K_0(\kappa R)\}\; K_{is}(\kappa R)}{R}\,\mathrm{d}R. \tag{3.2}$$

The precise *ansatz* used in (3.1) has been chosen with hindsight from study of the equivalent two-dimensional problem and a desire to retain as simple as possible a description in the interests of applicability. The method adopted here will be first to satisfy the surface condition (2.4) and then to examine the way in which the remaining asymptotic conditions may be satisfied. To satisfy (2.4) the value of $R^{-1}\phi_\theta$ on the SWL is first examined. Thus

$$R^{-1}\phi_\theta|_{\theta=0} = \int_0^\infty sA(s)\sinh s\alpha\, K_{is}(\kappa R)R^{-1}\,\mathrm{d}s \tag{3.3}$$

but noting the result §3.71 (1) in Watson (1944) and the fact that both $K_{is}(\cdot)$ and $A(s)$ are of even parity in s, the integral above is equal to

$$I = \frac{-\kappa}{2\mathrm{i}}\int_{-\infty+\mathrm{i}}^{+\infty+\mathrm{i}} A(s-\mathrm{i})\sinh(s-\mathrm{i})\alpha\, K_{is}(\kappa R)\,\mathrm{d}s.$$

Let S_δ^- denote the strip $-1 \leqslant \mathrm{Im}(s) \leqslant 0$ with the disk $|s+\mathrm{i}| < \delta, 0 \leqslant \mathrm{Arg}\,(s+\mathrm{i}) \leqslant \pi$ removed. Three conditions will now be assumed which can be examined *a posteriori*:

(C1) $A(s)$ is regular in S_δ^-;

(C2) $A(s)$ is meromorphic and has a simple pole at $s = -\mathrm{i}$ (with residue denoted χ_0);

(C3) $\mathrm{Lim}_{X\to\infty} \int_0^1 A(\pm X + \mathrm{i}y - \mathrm{i}) \exp(\alpha X) K_{iX-y}(\kappa R)\,\mathrm{d}y \to 0$.

It can thus be deduced from Cauchy's theorem that

$$I = \frac{-\kappa}{2\mathrm{i}} \mathrm{PV} \int_{-\infty}^{+\infty} A(s-\mathrm{i}) \sinh(s-\mathrm{i})\alpha\, K_{is}(\kappa R)\,\mathrm{d}s - \frac{\mathrm{i}\pi\kappa \sin\alpha}{2} \chi_0\, K_0(\kappa R) \tag{3.4}$$

where PV denotes the principal value. It is now required that

$$\left\{ A(s)\cosh s\alpha + \frac{\kappa}{\mathrm{i}} A(s-\mathrm{i}) \sinh \alpha(s-\mathrm{i}) \right\} = \Lambda(s), \tag{3.5}$$

where $\Lambda(s)$ is of *odd* parity. Replacing s by $-s$ and using the even parity of A, Λ can be eliminated from the pair of equations, leaving a single second-order functional equation,

$$C(p+1) - 2\mu\, C(p) \cot p\alpha - C(p-1) = 0, \tag{3.6}$$

where

$$s = \mathrm{i}p, \quad C(p) = \Xi(p) A(\mathrm{i}p) \sin p\alpha, \quad \Xi(p+1) = \Xi(p), \quad \mu = \kappa^{-1} (= \operatorname{cosec}\gamma) > 1. \tag{3.6a}$$

If A can be so constructed, it follows that (3.4) may be replaced by

$$I = \frac{1}{2} \int_{-\infty}^{+\infty} A(s)\cosh s\alpha\, K_{is}(\kappa R)\,\mathrm{d}s - \frac{\mathrm{i}\pi\kappa \sin\alpha}{2} \chi_0\, K_0(\kappa R) \tag{3.7}$$

having noted that the integral with Λ vanishes identically. It is now a straightforward matter to select the value

$$\lambda = -\tfrac{1}{2}\mathrm{i}\pi\kappa\,\chi_0 \sin\alpha \tag{3.8}$$

to ensure that condition (2.4) is satisfied. It remains therefore to solve the functional equation in such a way that the asymptotic conditions (2.5) and (2.6) are satisfied. Note that the arbitrary 1-periodic function Ξ may be even or odd, thus both even and odd solutions C will be required.

4. Solutions of functional equation

It is possible to apply a discrete Fourier transform in order to construct solutions of (3.6) for a discrete spectrum of beach slope α. Thus posit

$$C(s) = \rho^s \sum_{j=-\infty}^{\infty} a_j\, \mathrm{e}^{\mathrm{i}s\alpha j} \tag{4.1}$$

and substitute into (3.6), giving instead the general recurrence relation

$$a_{j-1} \left\{ \rho \mathrm{e}^{\mathrm{i}\alpha(j-1)} - 2\mathrm{i}\mu - \frac{1}{\rho} \mathrm{e}^{-\mathrm{i}\alpha(j-1)} \right\} = a_{j+1} \left\{ \rho \mathrm{e}^{\mathrm{i}\alpha(j+1)} + 2\mathrm{i}\mu - \frac{1}{\rho} \mathrm{e}^{-\mathrm{i}\alpha(j+1)} \right\}. \tag{4.2}$$

A requirement of compact support for the coefficients leads to $a_{-J-1} = a_{K+1} = 0$, $J, K \in \boldsymbol{N}$. Then the term in the left-hand side brace on the of (4.2) must vanish, when $j = K$ and, vice versa the term in the right-hand-side brace when $j = -J$. This gives the pair of equations

$$\rho \mathrm{e}^{\mathrm{i}\alpha(-J+1)} + 2\mathrm{i}\mu - \rho^{-1} \mathrm{e}^{-\mathrm{i}\alpha(-J+1)} = 0; \quad \rho \mathrm{e}^{\mathrm{i}\alpha(K-1)} - 2\mathrm{i}\mu - \rho^{-1} \mathrm{e}^{-\mathrm{i}\alpha(K-1)} = 0$$

so that, eliminating μ, it may be deduced that a solution for any incidence angle will be possible in this form provided that either

(i) $\alpha = \pi(1+2N)/(J+K-2)$, where $J+K>2, J, K, N \in \boldsymbol{N}$, or
(ii) $\rho^2 = \mathrm{e}^{\mathrm{i}\alpha(J-K)}$.

Case (ii) leads to edge waves so only case (i) is considered here. The compact support must be symmetric, i.e. $K = J$, or else the finite form of the expansion must be generalized in the case $\mu \geqslant 1$. The roots ρ (for general N) are

$$\rho_{1,2} = (-1)^N \mu \pm (\mu^2 - 1)^{1/2}.$$

Then, purely for brevity, restrict attention to cases where $N = 0$. This gives Stoker's (1958) slope angles $\alpha = \pi/2M$ for integer M. Define, for convenience, $\mu = \cosh\sigma\,(\sigma \text{ real})$, so that upon writing also $a_j = c_{(j+J-1)/2}$, a 'solution' of the recurrence relation may be expressed by

$$c_k = \frac{c_0 \cosh\sigma}{\cos(k\alpha - \mathrm{i}\sigma)\cos k\alpha} \prod_{j=1}^{k} -\cot j\alpha \cot(j\alpha - \mathrm{i}\sigma) \tag{4.3}$$

it being understood that the cosines are 'cancelled' when $k = J - 1\,(=\pi/2\alpha)$. A solution for $C(s)$ is given by

$$C(-\mathrm{i}s) = \mathrm{e}^{s(-\mathrm{i}\sigma - \frac{1}{2}\pi)} \sum_{j=0}^{J-1} c_j\, \mathrm{e}^{2j\alpha s} \tag{4.4}$$

and a second solution could be given by replacing σ by $-\sigma$, or, more simply, by noting that whenever $C(s)$ is a solution of (3.6), then so is $C(-s)$. In terms of the required even parity of $A(s)$ there follows, through the choice of the *i*-periodic function basis $Q{:}\{1, \coth\pi s\}$, the construction with arbitrary constants A, B,

$$A(s) = AA_+(s) + BA_-(s)\coth\pi s,$$

where $A_+(s)$ and $A_-(s)$ are of respectively even and odd parity given by

$$A_\pm(s) = [C(-\mathrm{i}s) \mp C(\mathrm{i}s)]\,/2\sinh s\alpha. \tag{4.5}$$

This will render A_- a simple zero at $s = 0$. To see this note that, in view of the construction (4.4), C will be finite at $s = \pm 1$ and from the equation (3.6) it follows that any even solution $C(-\mathrm{i}s) + C(\mathrm{i}s)$ must have a zero at $s = 0$ and the function being even, this must be a double zero. Thus, with this choice, $A(s)$ is integrable at $s = 0$.

Therefore the general solution can be written

$$\phi(R,\theta) = \lambda K_0(\kappa R) + \int_0^\infty \{AA_+(s) + BA_-(s)\coth\pi s\}\cosh s(\theta+\alpha)\, K_{is}(\kappa R)\,\mathrm{d}s \tag{4.6}$$

and, by considering full-depth near- and far-field asymptotics, it is noted that the particular asymptotic requirements (2.5) and (2.6) are now satisfied. The first of these is dealt with by the observation that the term in the brace in (4.6) is regular at $s = 0$, for this means that the integral may be replaced by

$$\frac{\pi}{2\mathrm{i}}\,\mathrm{PV}\int_{-\infty}^{\infty} \{AA_+(s) + BA_-(s)\coth\pi s\}\cosh s(\theta+\alpha)\,\mathrm{cosech}\,\pi s\, I_{-is}(\kappa R)\,\mathrm{d}s \tag{4.7}$$

which for small R is dominated by the residue arising from the simple pole at $s = 0$. With $I_0 \approx 1$, that result follows on observing that $\lambda K_0(\kappa R)$ has a logarithmic singularity as $R \to 0$. A discussion of the full near-field asymptotics is deferred to § 8.

Far-field asymptotics will be determined by the behaviour of the integrand as $s \to \infty$. Clearly,

$$A_{\pm}(s) \sim \mathrm{e}^{s(\frac{1}{2}\pi-\alpha)} \left\{\mathrm{e}^{-\mathrm{i}\sigma s} c_{J-1} \mp \mathrm{e}^{\mathrm{i}\sigma s} c_0\right\}. \tag{4.8}$$

In view of the exact result

$$\int_0^\infty 2\cosh s\left(\tfrac{1}{2}\pi - \alpha - \mathrm{i}\sigma\right)\,\cosh s(\theta+\alpha)\,K_{is}(\kappa R)\,\mathrm{d}s$$
$$= \Delta(\sigma) \equiv \pi\,\exp\{-\kappa R\,\sin(\alpha + \mathrm{i}\sigma)\,\cos(\theta+\alpha)\}\,\cosh\,\{\kappa R\cos(\alpha+\mathrm{i}\sigma)\,\sin(\theta+\alpha)\}$$

(see e.g. Oberhettinger 1972 p. 245), it is now an easy matter to rewrite the asymptotics for $A_+(s), A_-(s)$ with exponentials replaced by hyperbolic functions. From (4.6) there follows

$$\phi(R,\theta) \sim \{(A+B)c_{J-1}\}\,\Delta(\sigma) - \{(A-B)c_0\}\Delta(-\sigma).$$

The discussion in Appendix A illustrates why the dominant behaviour occurs on $\theta = 0$, the solution decaying with R at least algebraically on rays θ = negative constant. The result, from (4.6) with the term in the brace replaced by its surface asymptotics, is therefore

$$2\pi^{-1}\phi(R,0) \sim \mathrm{e}^{-\mathrm{i}R(1-\kappa^2)^{1/2}}\,\{(A+B)c_{J-1}\} - \mathrm{e}^{\mathrm{i}R(1-\kappa^2)^{1/2}}\,\{(A-B)c_0\}$$

and standing or progressing waves can be constructed as required. For the incoming progressing wave ϕ_∞, choose $A = -B = -1/\pi c_0$.

5. Perfectly reflected waves

It is well-known that, in the case of normal incidence, perfectly reflected waves, finite at the shoreline, are expressible in closed form for the special slope angles currently being considered. Writing $\beta_k = \exp(\mathrm{i}\pi(k/M + 1/2))$, the expansion given by Stoker (1958) is

$$\phi_r = \mathrm{Re}\left\{\sum_{k=1}^{M} c_k \exp(R\mathrm{e}^{\mathrm{i}\theta}\,\beta_k)\right\}, \tag{5.1}$$

where (purely for this expression)

$$c_k = \exp\left\{\mathrm{i}\pi\left(\frac{M+1}{4} - \frac{k}{2}\right)\right\}\prod_{j=1}^{k-1}\cot\left(\frac{j\pi}{2M}\right), \quad j > 1; \quad c_1 = \bar{c}_M.$$

This solution has been used extensively by other authors e.g. Keller (1961) investigating shoreline amplification, Blondeaux & Vittori (1995) studying the excitation of edge waves and more recently by Ehrenmark & Williams (1996) examining the efficiency of the mild-slope equation on a steep beach.

In the case of oblique wave attack, a perfectly reflected wave remains expressible in closed form. This observation appears to be less well-known and has gone largely unnoticed in some earlier descriptions of the classical problem (e.g. Peters 1952; Lauwerier 1959) although, of course, Ursell (1952) noted that, for edge wave motion under discrete frequencies, the solution which is finite at the origin is expressible as a sum of exponentials. It was Roseau (1952) who earlier showed that a generalization

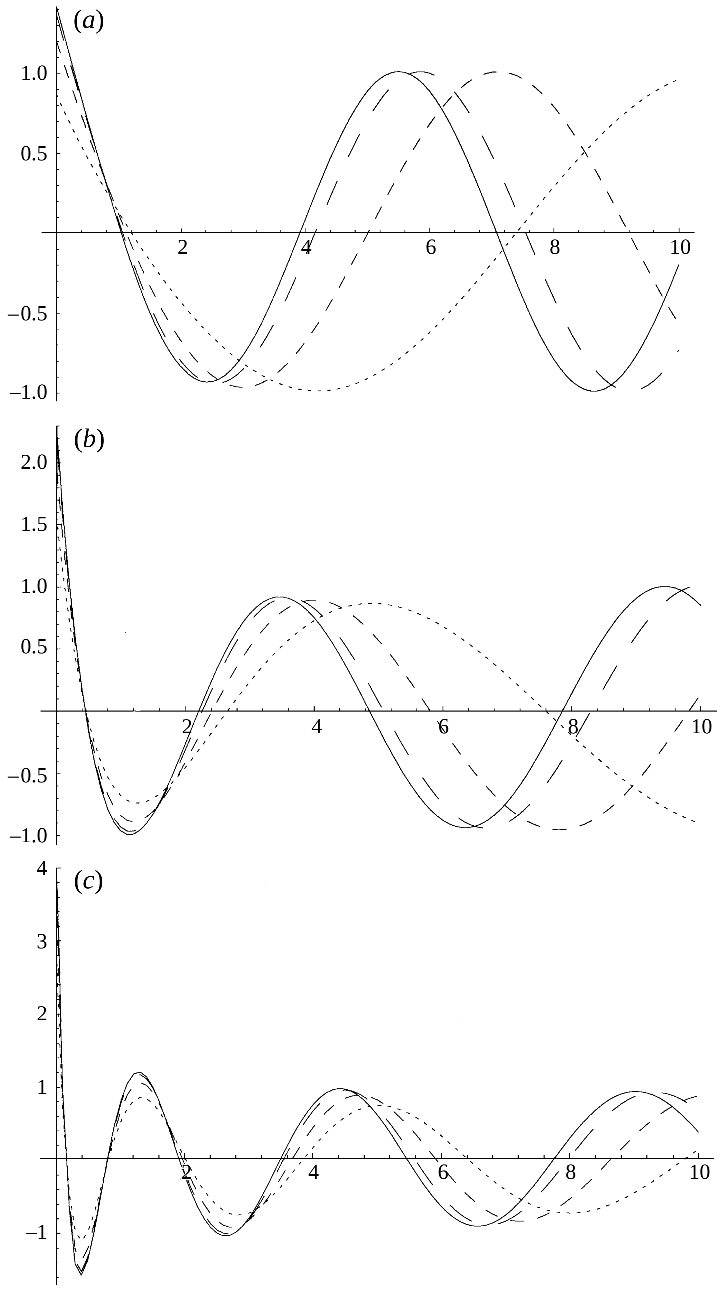

FIGURE 2 (*a*–*c*). For caption see facing page.

to expansion (5.1), in the case of Stoker's slope angles, was possible and later (1958) that this type of expression for edge wave motion was also possible for a continuous spectrum of frequencies but then only for special parameters.

Some details of the present discussion are deferred to Appendix B but it may be noted here that the result depends only on showing that in (4.7) the expression for A_+ may be reduced to a hyperbolic polynomial in terms of s, so that exact K–L inversion is possible.

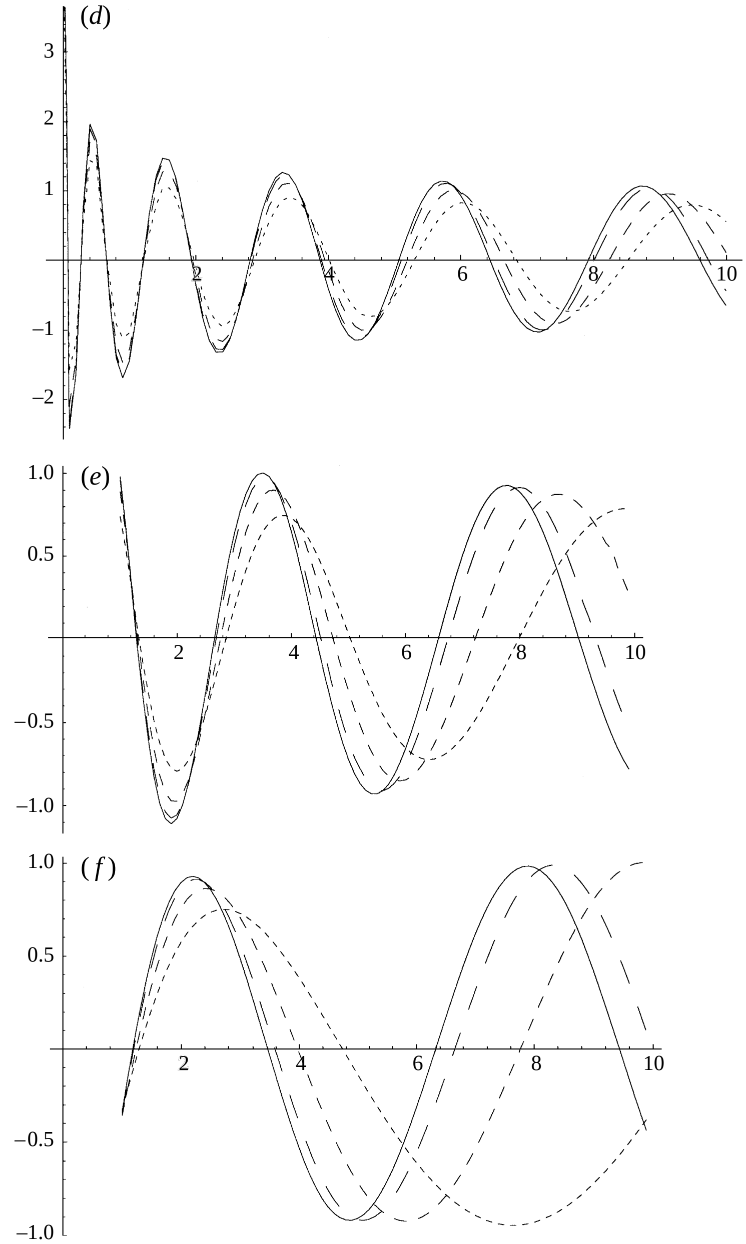

FIGURE 2. Potentials for oblique wave attack: (*a*–*d*) Regular wave: (*a*) 45° beach, (*b*) 18° beach, (*c*) 6° beach, (*d*) 2° beach; (*e*–*f*) Singular wave: (*e*) 6° beach, (*f*) 18° beach. Full line, incidence angle 1°; broken line, incidence angle 21°; dashed line, incidence angle 41°; dotted line, incidence angle 61°.

The result, from Appendix B, is

$$\{A_+, A_-\} = 2\sum_{r=1}^{J-1} d_r \,\{\cosh,\, -\sinh\}\; s\left(\tfrac{1}{2}\pi - (2r-1)\alpha + \mathrm{i}\sigma\right), \tag{5.2}$$

where $d_r = \sum_{j=r}^{J-1} c_j$. This facilitates the K–L inversion, with the result that the potential ϕ_+ of the solution which is finite at the origin may be written

$$\begin{aligned}\phi_+ &= 2\sum_{r=1}^{J-1} d_r \int_0^\infty K_{is}(\kappa R)\cosh s(\theta+\alpha)\cosh s\left(\tfrac{1}{2}\pi-(2r-1)\alpha+\mathrm{i}\sigma\right)\mathrm{d}s \\ &= \tfrac{1}{2}\pi\sum_{r=1}^{J-1} d_r\left\{\mathrm{e}^{(-\kappa R\,\sin[(2r-1)\alpha-\mathrm{i}\sigma-(\theta+\alpha)])}+\mathrm{e}^{(-\kappa R\,\sin[(2r-1)\alpha-\mathrm{i}\sigma+(\theta+\alpha)])}\right\}.\end{aligned} \tag{5.3}$$

Similarly a fundamental solution ϕ_- which has the logarithmic singularity at the origin may be given by

$$\begin{aligned}\mathrm{i}\phi_- &= \lambda K_0(\kappa R)-2\sum_{r=1}^{J-1} d_r\int_0^\infty K_{is}(\kappa R)\coth\pi s\cosh s(\theta+\alpha)\sinh s\left(\tfrac{1}{2}\pi-(2r-1)\alpha+\mathrm{i}\sigma\right)\mathrm{d}s \\ &= \lambda K_0(\kappa R)-2\sum_{r=1}^{J-1} d_r\int_0^\infty K_{is}(\kappa R)\sinh s(\theta+\alpha)\sinh s\left(\tfrac{1}{2}\pi-(2r-1)\alpha+\mathrm{i}\sigma\right)\mathrm{d}s \\ &\quad -2\sum_{r=1}^{J-1} d_r\int_0^\infty K_{is}(\kappa R)\frac{\cosh s(\pi-\theta-\alpha)\sinh s\left(\tfrac{1}{2}\pi-(2r-1)\alpha+\mathrm{i}\sigma\right)}{\sinh\pi s}\mathrm{d}s.\end{aligned} \tag{5.4}$$

As an example, for the 45° beach, $J=3$ and we get, from (5.3) the well-known result

$$\phi_+ = \mathrm{e}^{y}(T\,\cos\,Tx-\sin\,Tx)+\mathrm{e}^{-x}(T\,\cos\,Ty+\sin\,Ty),$$

where $(x,y)=R(\cos\,\theta,\,\sin\,\theta)$, $T=\tanh\sigma$. The special case $\alpha=\frac{1}{6}\pi$ ($J=4$) has also been written in Appendix B.

It is remarked that the similarity in structure between (5.3) and the result of Roseau (1952) for the same problem conceals the effort required to bring these two solutions together in the general case. A wave of unit amplitude at infinity can always be obtained by appropriate choice of one of the constants c_k. Results of computing ϕ_+ and ϕ_- (which are both real valued) are displayed in figure 2 for a range of beach slopes and incidence angles.

6. Progressing waves against a vertical cliff

The case $\alpha=\frac{1}{2}\pi$ is examined explicitly. For this case, $N=0$, $J=K=2$ and hence $a_{-1}=a_1$. With arbitrary reflection at the shoreline there follows

$$A(s) = A\,\cos\,\sigma s + B\,\sin\,\sigma s\coth s\pi. \tag{6.1}$$

This problem was originally solved by Weinstein (1949) but for convenience it is remarked that a full exposition of the method of solution (which is different from that of Peters (1952) for the more general problem) is given by Stoker (1958). It is shown here that this solution is easily recovered in the present description by using (6.1) as the brace terms in (4.6). The part corresponding to $A=1$, $B=0$ gives the standing wave

$$\phi_+ = \tfrac{1}{2}\pi\mathrm{e}^{R\,\sin\,\theta}\,\cos\,[R\,(1-\kappa^2)^{1/2}\,\cos\,\theta]$$

whilst that corresponding to $A = 0$, $B = 1$ gives the other standing wave

$$\phi_- = \tfrac{1}{2}\pi e^{R\sin\theta}\sin[R(1-\kappa^2)^{1/2}\cos\theta] - \frac{i\pi\kappa}{2}\chi_0 K_0(\kappa R) + \int_0^\infty \frac{\cosh s(\frac{1}{2}\pi-\theta)\sin\sigma s K_{is}(\kappa R)}{\sinh(\pi s)}\,ds.$$

In this, χ_0 is the residue at $s = -i$ of $\sin\sigma s \coth s\pi$. Then if $\theta = 0$ there follows

$$\phi_2(R,0) = \tfrac{1}{2}\pi\sin[R(1-\kappa^2)^{1/2} - \frac{(1-\kappa^2)^{1/2}}{2}K_0(\kappa R) + \frac{1}{2}\int_0^\infty \frac{\sin\sigma s K_{is}(\kappa R)}{\sinh(\pi s/2)}\,ds.$$

On noting the exact result (see Oberhettinger, 1972 p. 245)

$$\int_0^\infty \frac{\sin\sigma s K_{is}(\kappa R)\,ds}{\sinh(\pi s/2)} = \sinh\sigma\int_0^\infty \exp(-t\cosh\sigma)K_0[(\kappa^2R^2+t^2)^{1/2}]\,dt$$

and replacing the Macdonald functions by Hankel functions, Weinstein's solution (as given by Stoker) is recovered to within a multiplicative constant.

7. Progressing waves attacking a beach

The general case: $\alpha = \pi/2M$ is examined. For ϕ_+ in closed form the alternative partitioning

$$\phi_+ = \tfrac{1}{2}\pi\left\{d_1 e^{\kappa R\sin(\theta+i\sigma)} + d_{J-1}e^{\kappa R\sin(\theta-i\sigma)}\right\} + \tfrac{1}{2}\pi\sum_{r=1}^{J-2} e^{-R\sin(2r\alpha-\theta)}\left\{d_{r+1}e^{iR\tanh\sigma\cos(2r\alpha-\theta)} + d_{J-1-r}e^{-iR\tanh\sigma\cos(2r\alpha-\theta)}\right\} \quad (7.1)$$

exposes the dominant asymptotic terms as $R \to \infty$ in the first brace. Note from (4.3) that $|c_{J-1}| = |c_0|$ so that (with $\operatorname{Arg} c_0$ arbitrary) taking $c_{J-1} = -\bar{c}_0$ (where the bar denotes a complex conjugate) there follows $d_r = \bar{d}_{J-r}$ and hence the alternative expression

$$\phi_+ = \pi\,\mathrm{Re}\left[\sum_{r=1}^{J-1} d_{J-r}e^{-\kappa R\sin(2(r-1)\alpha-\theta+i\sigma)}\right].$$

Similarly, (5.4) can be simplified by inverting the first of the summations to yield

$$i\phi_- = \lambda K_0(\kappa R) - i\pi\,\mathrm{Im}\left[\sum_{r=1}^{J-1} d_{J-r}e^{-\kappa R\sin(2(r-1)\alpha-\theta+i\sigma)}\right] - 2\sum_{r=1}^{J-1} d_r\int_0^\infty K_{is}(\kappa R)\frac{\cosh s(\pi-\theta-\alpha)\sinh s\left(\frac{1}{2}\pi-(2r-1)\alpha+i\sigma\right)}{\sinh\pi s}\,ds, \quad (7.2)$$

where λ is given by $\lambda = \kappa\sin\alpha\sum_{r=1}^{J-1} d_r\cos((2r-1)\alpha - i\sigma)$. It is now a straightforward matter to construct waves of arbitrary phase using suitable combinations of the fundamental basis $\{\phi_+, \phi_-\}$. The asymptotic form at infinity stems from the $r = 1$ term in the first of the sums whilst the integrals in (7.2) converge at least like $\exp(-2\alpha s)$ on the surface and are easily computed following the methods described by Ehrenmark (1995). Note that $d_1 = \overline{d_{J-1}} = \overline{c_{J-1}}$. The incoming progressing wave Φ_P is therefore constructed by the combination $\phi_+ - i\phi_-$ giving

$$\Phi_P = \mathrm{Re}\{[\phi_+ - i\phi_-]\exp[-i(t+\kappa z)]\} \quad (7.3)$$

as the wave which has the asymptotic behaviour $\pi\,|c_{J-1}|\,\cos\left\{t+\kappa z+R(1-\kappa^2)^{1/2}-\operatorname{Arg}(c_{J-1})\right\}$. To identify $\operatorname{Arg}(c_{J-1})$, note that, setting $\operatorname{Arg}(c_0)=\beta$, it follows that $\operatorname{Arg}(c_{J-1})=\pi-\beta$ and hence that

$$\operatorname{Arg}\left(\frac{c_{J-1}}{c_0}\right)=\sum_{j=1}^{J-1}\operatorname{Arg}\left\{\frac{\cos(2j\alpha+\mathrm{i}\sigma)-\cos(\mathrm{i}\sigma)}{-\cos(2j\alpha-\mathrm{i}\sigma)+\cos(\mathrm{i}\sigma)}\right\}=\pi-2\beta,$$

where $\beta=\pi J/2-\sum_{j=1}^{J-1}\tan^{-1}(\tanh\sigma\cot j\alpha)$. One of the main aims is to write the full solution in such a way that Stoker's (1958) potential functions are recovered in the limit $\sigma\to\infty$. To this end, it is noted that $\beta\to\pi(J+2)/4$ in this limit. Thus $\operatorname{Arg}(c_{J-1})\to\pi(2-J)/4$ from which the limiting asymptotic form of ϕ_+ is seen to be

$$\lim_{\sigma\to\infty}\phi_+=\pi\,|c_{J-1}|\,\cos(R+\pi(n-1)/4)),$$

where $n=J-1$, thus in exact agreement with Stoker (1958, p. 84) since the amplitude to be assigned is arbitrary.

Corollary. *Assume, for the moment that* $J=2N+1$. *If the amplitude of* ϕ_+ *is set to unity at infinity, then the shoreline amplitude* $\phi_+(0)$ *is given by*

$$\phi_+(0)=\sum_{k=1}^{2N}\mathrm{d}_k=\sum_{k=0}^{N-1}(N-k)(c_{2N-k}-c_k)=2\,\mathrm{Re}\sum_{k=0}^{N-1}(N-k)c_{2N-k}=2\mathrm{Re}\sum_{k=1}^{N}kc_{N+k}$$

since $\sum_{j=0}^{2N}c_j=0$ *and* $c_k=-\overline{c_{2N-k}}$.

Consider the limiting form of this as $\sigma\to\infty$. Under this limit, (4.3) can be written

$$c_{N+k}=\frac{c_0\,\mathrm{e}^{-\mathrm{i}(N+k)\alpha}}{\cos(N+k)\alpha}\prod_{j=1}^{N+k}\mathrm{e}^{-\mathrm{i}\pi/2}\cot j\alpha=\frac{c_0\,\mathrm{e}^{-\mathrm{i}(N+k)(\alpha+\frac{1}{2}\pi)}}{\cos(N-k)\alpha}\prod_{j=1}^{N-k}\cot j\alpha$$

from which, noting that $(N+k)(\alpha+\frac{1}{2}\pi)-\operatorname{Arg}(c_0)=\frac{1}{2}\pi-k(\alpha+\frac{1}{2}\pi)$, there follows

$$\lim_{\sigma\to\infty}\phi_+(0)=2\sum_{k=1}^{N}k\,\frac{\sin k(\alpha+\frac{1}{2}\pi)}{\cos(N-k)\alpha}\prod_{j=1}^{N-k}\cot j\alpha$$

taking the product as unity when $k=N$. It is shown in Appendix C that this expression is equivalent to $(2N)^{1/2}$. Moreover, it is well-known in the two-dimensional problem (see e.g. Keller 1961), that the shoreline amplification factor is equal to this value. In the case of perfectly reflected waves under oblique incidence we can see from the definition of Φ (§2) that motion is progressing in the longshore (z) direction with a wavenumber proportional to the sine of the angle of incidence γ. Local amplification factors will therefore vary from 0 to a bed-slope and incidence-angle-dependent maximum, the value at any time depending on the phase of the longshore wave motion. However, this maximum is readily seen to be $\phi_+(0)$. The idea is illustrated here for the case $J=2N+1$; the case $J=2N$ may be similarly treated.

Shown, in figure 3, are curves for the shoreline amplification, for various beaches, plotted as a function of incidence angle. It is noted that, for certain slopes, this value approaches 1 as the incidence is gradually made more glancing. These are the 'critical slopes' ($\alpha=\pi/(4n+2)$, $n\in\boldsymbol{N}$) at which new modes of the Ursell (1952) edge waves appear; the so-called *cut-off* modes. These modes therefore have the same amplitude

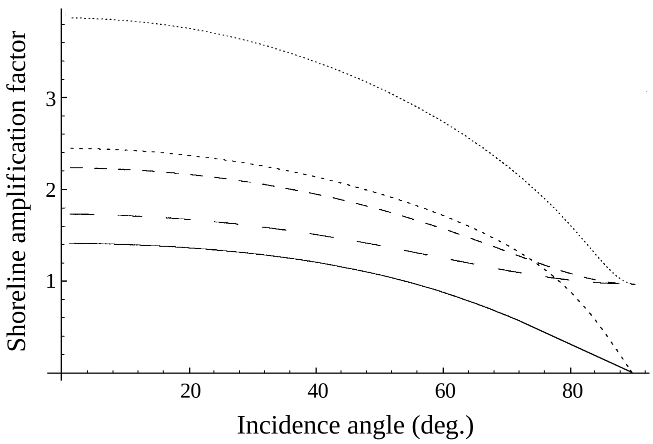

FIGURE 3. Maximum shoreline amplification factor as a function of incidence angle. Full line, beach slope 45°; broken line, beach slope 30°; dashed line, beach slope 18°; dotted line, beach slope 15°; finely dotted line, beach slope 6°.

at infinity as at the shoreline. For the set ($\alpha = \pi/4n$, $n \in \boldsymbol{N}$) the limiting value is instead zero, which observation Whitham (1979) explained by noting that 'incoming' and 'reflected' waves are exactly out of phase with $\mathrm{Arg}\big(c_{J-1}/c_0\big) = \pi(1-J)$ as $\sigma \to 0$ from the above relation. In §4 the possibility was noted of solving (4.2) under the alternative assumption $\rho^2 = \mathrm{e}^{\mathrm{i}\alpha(J-K)}$. This will yield the edge waves discussed very fully by Ursell (1952) and, albeit with a slightly different ansatz, by Roseau (1958). It may be noted, in passing, that the present description (like that of Whitham 1979) is sufficiently general to include these waves. The more (shoreline) algebraically unbounded 'further' solutions discussed by Roseau (1958) are not described here however.

The section is completed by summarizing a full form of an incoming progressing wave at angle of incidence $\gamma(=\sin^{-1}\kappa)$. By writing $\varsigma = t + \kappa z$, and $Z = 1/\left|c_{J-1}\right| \sum_{r=1}^{J-1} d_{J-r} \mathrm{e}^{-\kappa R \sin(2(r-1)\alpha-\theta+\mathrm{i}\sigma)}$, the potential (taken to be of unit amplitude at infinity) may be written

$$\Phi_P = \mathrm{Re}[Z\mathrm{e}^{\mathrm{i}\varsigma}] + \mathrm{i}\sin\varsigma\left\{\lambda K_0(\kappa R) - 2\sum_{r=1}^{J-1} d_r \int_0^\infty K_{is}(\kappa R)\frac{\cosh s(\pi-\theta-\alpha)\sinh s\vartheta_r}{\sinh \pi s}\,\mathrm{d}s\right\}, \tag{7.4}$$

where

$$\cosh\sigma = \kappa^{-1}, \quad \lambda = \kappa\sin\alpha\sum_{r=1}^{J-1} d_r \sin\vartheta_r, \quad \vartheta_r \equiv \tfrac{1}{2}\pi - (2r-1)\alpha + \mathrm{i}\sigma, \quad d_r = \sum_{j=r}^{J-1} c_j,$$

and the constants c_k satisfy

$$c_k = \frac{-c_0\cosh\sigma}{\sin(k\alpha-\mathrm{i}\sigma)\sin k\alpha}\prod_{j=1}^{k-1} -\cot j\alpha\cot(j\alpha-\mathrm{i}\sigma), \quad \sum_{j=0}^{J-1} c_j = 0, \quad c_k = -\bar{c}_{J-1-k}.$$

8. Near-field expansion

One of the specific shortcomings of previous models has been a lack of detail of the solution in the nearfield. None of the authors Peters (1952), Roseau (1952), Lauwerier (1959) or Whitham (1979) add to the observation that one of the fundamental components has a logarithmic singularity at $R = 0$. However, the usefulness of a full expansion has been demonstrated by the present author (1996) and by Ehrenmark & Williams (1996) who examined the characteristic behaviour of a mild-slope equation by direct comparison with such an expansion.

The simplest way to obtain a near-field expansion is to rewrite the solution integral with the modified Bessel function as argument and to use the residue theorem in the upper half-plane. This procedure has been described elsewhere in this work and, with the help of the asymptotics in Appendix A to verify the procedure, it is relatively straightforward. To do this, write ϕ_- from the expressions (5.3), (5.4) in the form

$$\mathrm{i}\phi_- = \lambda K_0(\kappa R) + \mathrm{i}\pi \sum_{r=1}^{J-1} d_r \, \mathrm{PV} \int_{-\infty}^{\infty} \sinh s\vartheta_r \coth \pi s \cosh s(\theta + \alpha) \operatorname{cosech} \pi s \, I_{-is}(\kappa R) \, \mathrm{d}s.$$

The principal value integral contributes a 'half' residue at $s = 0$ and full residues at the other poles in the upper half-plane. Writing $\sum_{res}$ to denote these, there follows

$$\mathrm{i}\phi_- = \lambda K_0(\kappa R) - \sum_{r=1}^{J-1} d_r \left\{ \vartheta_r I_0(\kappa R) + 2\pi^2 \sum_{res} \sinh s\vartheta_r \coth \pi s \cosh s(\theta + \alpha) \operatorname{cosech} \pi s \, I_{-is}(\kappa R) \right\}.$$

Note that remaining residues now arise from double poles and the resulting expansion is

$$\phi_- = \sum_{r=1}^{J-1} d_r \, \phi_-^{(r)},$$

where

$$\mathrm{i}\phi_-^{(r)} = \lambda_r K_0(\kappa R) - \sum_{k=0}^{\infty} (-)^k \mu_k \Big\{ [\vartheta_r \cos k\vartheta_r \cos k(\theta + \alpha) - (\theta + \alpha) \sin k\vartheta_r \sin k(\theta + \alpha)] I_k(\kappa R) + \sin k\vartheta_r \cos k(\theta + \alpha) \frac{\partial I_\nu(\kappa R)}{\partial \nu}\Big|_{\nu = k} \Big\}, \tag{8.1}$$

$\lambda_r = \kappa \sin \alpha \cos((2r - 1)\alpha - \mathrm{i}\sigma)$ and $\mu_k = 1$, if $k = 0$, $= 2$ if $k > 0$. Whilst the convergence of the above series is uniform for all R, it is obvious (from the asymptotics of the modified Bessel function for $\kappa R \gg 1$) that practical calculation requires that $\kappa R \approx O(1)$.

A similar expression is easily written from (5.3) for ϕ_+, namely

$$\phi_+ = -\mathrm{i}\pi \sum_{r=1}^{J-1} d_r \, \mathrm{PV} \int_{-\infty}^{\infty} \cosh s\vartheta_r \cosh s(\theta + \alpha) \operatorname{cosech} \pi s \, I_{-is}(\kappa R) \, \mathrm{d}s$$

and the poles are now simple, giving the expansion

$$\phi_+ = \pi \sum_{r=1}^{J-1} d_r \left\{ I_0(\kappa R) + 2 \sum_{k=1}^{\infty} (-)^k \cos k\vartheta_r \cos k(\theta + \alpha) I_k(\kappa R) \right\}. \tag{8.2}$$

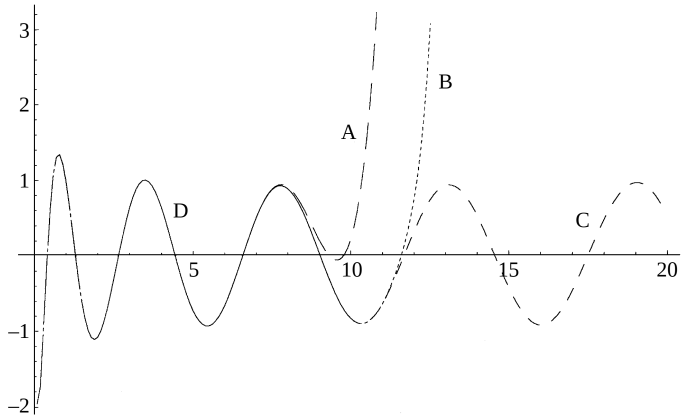

FIGURE 4. Numerical behaviour of near-field expansion of singular wave potential (from equation (8.4)). Case of 6° beach with near-normal (1°) wave incidence: Broken line A, 16 terms of expansion; dotted line B, 32 terms of expansion; dashed line C, 72 terms of expansion; full line D, exact solution from two-dimensional theory shown for $R < 10$.

Expanding the Cauchy products, a Taylor expansion of ϕ_+ is easily written:

$$\phi_+ = \pi \sum_{\rho=0}^{\infty} a_\rho (\kappa R/2)^\rho, \tag{8.3}$$

where

$$a_\rho = \sum_{r=1}^{J-1} d_r \sum_{j=0}^{[\rho/2]} \frac{1}{j!} \frac{e_{\rho-2j}}{(\rho-j)!}; \quad e_k = (-)^k \mu_k \cos k\vartheta_r \cos k(\theta+\alpha).$$

For the singular solution the Cauchy product treatment yields

$$\mathrm{i}\phi_- = -\log\left(\frac{\kappa R}{2}\right) \sum_{\rho=0}^{\infty} A_\rho \left(\frac{\kappa R}{2}\right)^\rho + \sum_{\rho=0}^{\infty} B_\rho \left(\frac{\kappa R}{2}\right)^\rho, \tag{8.4}$$

where, after writing $\Theta_r(k) = \vartheta_r \cos k\vartheta_r \cos k(\theta+\alpha) - (\theta+\alpha) \sin k\vartheta_r \sin k(\theta+\alpha)$, there follows

$$A_\rho = \sum_{r=1}^{J-1} d_r \sum_{j=0}^{[\rho/2]} \frac{1}{j!} \frac{{e'}_{\rho-2j}}{(\rho-j)!},$$

$$B_\rho = \sum_{r=1}^{J-1} d_r \sum_{j=0}^{[\rho/2]} \frac{1}{j!} \frac{\{{e'}_{\rho-2j}\,\psi(\rho-j+1) - (-)^\rho \mu_{\rho-2j} \Theta_r(\rho-2j)\}}{(\rho-j)!}$$

having set ${e'}_0 = \lambda_r$; ${e'}_k = (-)^k \mu_k \sin k\vartheta_r \cos k(\theta+\alpha)$, $k > 0$. In the above, ψ is the usual digamma function. The accuracy of the near-field expansion is seen in figure 4. Shown there, for ϕ_- are the approximations obtained by respectively 16, 32 and 72 terms of the above series for a limiting small incidence angle (in this case $\gamma = 1°$) which can therefore be compared with the known exact solutions for two-dimensional motion. With the curves coinciding, for modest values of R, it is not possible to see there precisely the accuracy obtained. As an example it is therefore recorded that, for the 6° beach singular potential at $R = 1$ the current expansion yields the value 0.976 889 for 'almost two-dimensional incidence' whereas Ehrenmark & Williams

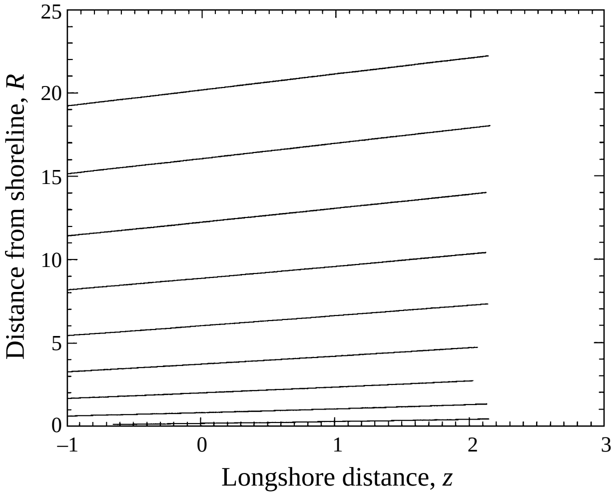

FIGURE 5. Refracted wave fronts for a progressing wave at 45° incidence on a beach of 6° slope.

(1996) noted 0.9769 from the evaluation of the solution integral for two-dimensional incidence.

As a further demonstration of the solution, refracted wave fronts have been calculated for the case $\alpha = 6°$, with an incidence angle of 45°. This involves the progressing wave construction (7.3), tabulating ϕ_+ and ϕ_- (from the above expansions) and then solving e.g. $\kappa z = \tan^{-1}\{(\phi_- - T\phi_+)/(T\phi_- + \phi_+)\}$, where $T = \tan \omega t$, for the wave crests (see e.g. Krauss 1973 p. 145). A tabulation on $R = [0.1(0.1)20]$ is obtained in a matter of minutes on a desktop computer. The result is shown graphically in figure 5. Readers interested in refraction may wish to compare this with the more approximate conventional refraction diagram usually obtained by estimating phase velocities from Airy theory and invoking a Fermat principle for wave rays. Note therefore, in figure 6 (a, b) the behaviour of refracted wave rays for increasingly oblique incidence. These are computed from the wave ray vector, given through e.g. (7.3) by

$$k = \nabla(\tan^{-1}(\phi_-/\phi_+) + \kappa z).$$

For the gentler 6° beach (figure 6 b) the arc CD shows the equivalent 'Airy theory' curve computed with the full expression for celerity $((g/k)\tanh kh)^{1/2}$ whilst the arc AB shows a computation using the 'shallow water celerity $(gh)^{1/2}$. The curves are started from data at D and B respectively. The computation from (8.5) appears identical to arc CD at this incidence of 60°. Instability is evident from (depending on beach slope) between 70° and 80° indicating a limitation of the theory. Peregrine & Ryrie (1983) demonstrated, with a nonlinear model, the phenomenon of anomalous refraction at the higher incidence angles and this appears to be consistent with instablility of computed results in a linear model. The more basic Airy model does not display the instability and it is possible that a fuller understanding of the asymptotics of both bounded and unbounded edge waves may throw some light on the interpretation of the instability in the present context. Note that some of the data (for 60° and 70° incidence) displayed by Peregrine & Ryrie (1983, figure 1) are extrapolated and reproduced, for comparison, in figure 6 (b) from which it is clear that agreement between linear and nonlinear models is better at the less oblique incidence angle.

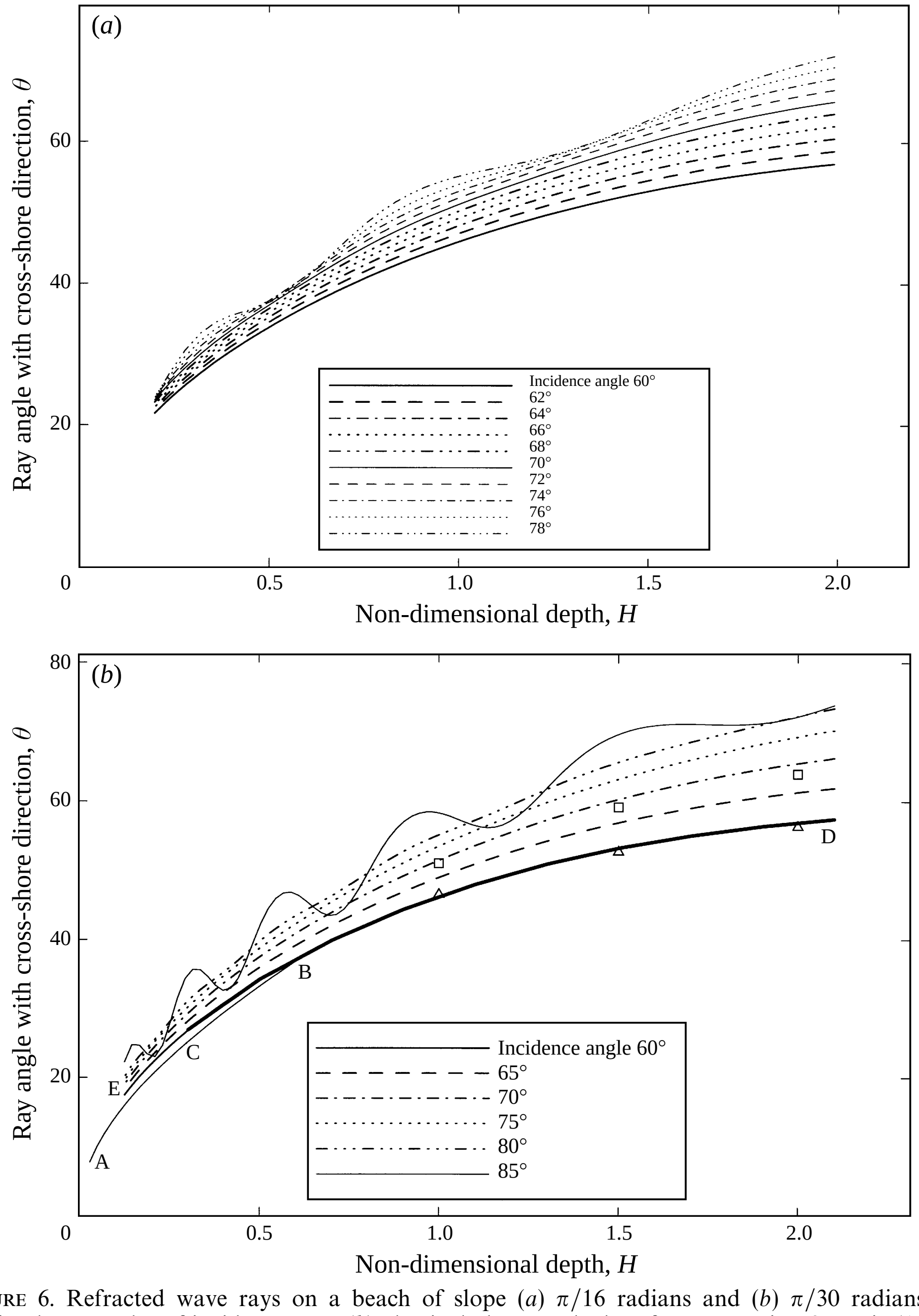

FIGURE 6. Refracted wave rays on a beach of slope (*a*) $\pi/16$ radians and (*b*) $\pi/30$ radians for varying large angles of incidence. Part (*b*) also includes sample data from Peregrine & Ryrie (1983): triangle, 60° incidence; square, 70° incidence. For ray at 60° incidence note: (i) Arc AB: shallow water Airy theory result; (ii) Arc CD (thick line): full Airy theory result; (iii) Arc ED (partly hidden line): present theory.

9. Set-down computations

As an application of the full theory, the computation of 'set-down' may be considered. This is the depression below SWL of the mean surface before wave breaking which in turn induces 'set-up' (Longuet-Higgins & Stewart 1963). In a frictionless theory, the latter cannot be modelled, but the former is due to the shoaling effect and the author has previously computed set-down (Ehrenmark 1994) in the normal incidence problem and found excellent agreement with the classical results of Longuet-Higgins & Stewart (1963) which were valid only for gentle slopes. The reader is referred to these works for a discussion of the physical assumptions. The fundamental expansions are taken in the usual form of asymptotic expansions in powers of a small ordering parameter ε which is representative of wave slope at large distances:

$$\{\phi, \eta\} = \varepsilon\{\phi_1, \eta_1\} + \varepsilon^2\{\phi_2, \eta_2\} + \dots .$$

The expansion, to second order, of velocity potential and wave height then yields a surface boundary condition which, when the mean over a wave period is taken, may be written

$$\langle\eta_2\rangle + \left\langle \eta_1 \frac{\partial^2 \phi_1}{R\partial t \partial \theta} \right\rangle = \left\langle -\tfrac{1}{2}(\nabla\phi_1)^2 \right\rangle$$

(see e.g. Blondeaux & Vittori 1995) where $\langle a \rangle$ denotes that mean of a and all quantities are evaluated on the SWL $\theta = 0$. From the surface conditions satisfied by the first-order quantities, the expression may be simplified to

$$\langle\eta_2\rangle = -\frac{1}{2}\left\langle \left(\frac{\partial\phi_1}{\partial R}\right)^2 + \left(\frac{\partial\phi_1}{\partial z}\right)^2 - \phi_1^2 \right\rangle\Bigg|_{\theta=0}. \tag{9.1}$$

The unreflected progressing wave is given by equation (7.3). It may be noted that both ϕ_+ and ϕ_- are real valued (following the conjugacy conditions on the constants d_r), so that taking $\phi_1 = \Phi_P$,

$$\phi_1 = \phi_+ \cos(t + \kappa z) + \phi_- \sin(t + \kappa z)$$

and hence

$$2\left\langle \left(\frac{\partial\phi_1}{\partial R}\right)^2 + \left(\frac{\partial\phi_1}{\partial z}\right)^2 - \phi_1^2 \right\rangle\Bigg|_{\theta=0} = (\phi_+')^2 + (\phi_-')^2 + (\kappa^2 - 1)\{\phi_+^2 + \phi_-^2\} \tag{9.2}$$

where the prime denotes an R-derivative. Computation is easily accomplished by differentiation of the series expansions given in § 8. This has been carried out for a beach of slope 30° and the results are shown in figure 7 and may be compared with the result developed by the author (1996) for normal incidence. Note, in particular, that an observer moving shoreward encounters a region of set-up prior to the main set-down. This is due to the trapped standing wave that is induced by the shoaling; the deep-water asymptotics of the two principal components ϕ_+ and ϕ_- being somewhat different. The square brackets in (7.1) and (7.2) denote the respective 'Airy' components whose contribution to (9.2) will vanish when aggregated, in the same way that Airy waves that 'do not feel the bottom' would have a zero set-down. The set-down due to a standing wave $\cos x \cos t$ is however a multiple of $\cos 2x$ (Krauss 1973) and in this way, the residual component of (7.2) will effectively contribute an additional standing wave albeit of seaward decreasing amplitude. The effect of this is therefore to induce alternately set-down and set-up in a region sufficiently far from the shore. This region migrates seaward as the beach slope decreases (see Ehrenmark 1994 for a fuller discussion).

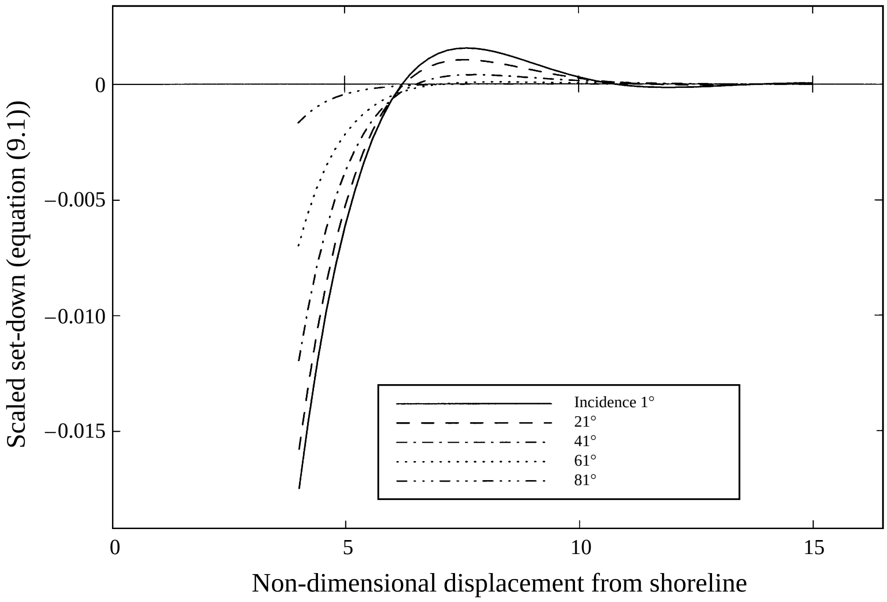

FIGURE 7. Set-down calculations for a beach of slope 30° calculated from equation (9.1) for varying angles of incidence.

10. Summary

A classical problem has been reviewed with the particular objective of obtaining a simple formulation which is readily adapted for computation. This contrasts with previous works where, with the exception of the case of the bounded perfectly reflected wave, methods of accurate evaluation have not been considered. Shallow beach calculations are generally made with some type of mild-slope approximation but the present theory, being essentially non-hydrostatic, is applicable to beaches of arbitrary slope and in particular may be of interest to sea-defence modellers or those working with flow over steep shingle beaches. One way the present model may be exploited is in the accurate assessment and calibration of the wave kinematic 'black box' used as an element in the computational flow chart by modellers working over variable topography. The validity of mild-slope approximations can be squeezed to extraordinary slopes (e.g. Booij 1983; Ehrenmark & Williams 1996) and the latter work has shown, in two dimensions, how modification of wave and phase velocities can further improve the performance of the approximated system even on a beach of slope 45°. This is done by comparison with the more 'exact' non-hydrostatic classical solution. The opportunity now exists for similar 'tuning' in a three-dimensional environment and those interested in testing and refining parabolic models of wave transformation should find the calculations herein of some value.

The refraction diagram and the set-down computations confirm that the present model works well outside the breaker zone. However, even in the case of non-breaking waves, there remains the difficulty of the logarithmically infinite values of potential at the shoreline. In reality this energy 'sink' is spread out through the dissipation zone – whether by viscous forces, bottom friction or by breaking events – but its effect in the far field remains the same, namely that the reflected wave is only of limited importance. Earlier work by the author in two-dimensions (1991) showed that (eddy) viscosity treatment is possible and in this way bounded solutions representing progressing waves in the far field may be constructed. The joining of the near- and far-field approximations currently needs to be done heuristically however, and a desirable aim would be a more systematic approach to finding a uniformly valid viscous solution. If this can be done there should be no reason why a similar approach cannot be made in the three-dimensional case using the present formulation as the basic solution. Moreover, it was shown in Ehrenmark (1991) how one effect of viscosity was to induce an element of standing wave behaviour in a shoreward progressing wave. This phenomenon will have implications for refraction also and it would be of considerable interest to examine the degree of 'turning' that can be obtained with an eddy viscosity model and whether the frictional element of this turning opposes or (as one might expect) reinforces that due to the shoaling.

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Appendix A

This Appendix details various asymptotic forms required in the main text.

(i) The asymptotics of $K_{is}(z)$ for $|s| \gg 1$, $z = o(|s|^{1/2})$, are controlled in the upper half-plane by that of $I_{-is}(z)$ and in the lower half-plane by that of $I_{is}(z)$. The first term of the power series expansion (Watson 1944 p. 79) in increasing powers of z is sufficient to determine the dominant asymptotic term for large $|s|$. The full expansion is

$$I_{-is}(z) = \sum_{m=0}^{\infty} \frac{\left(\frac{1}{2}z\right)^{-\mathrm{i}s+2m}}{m!\Gamma(1+m-\mathrm{i}s)} \tag{A 1}$$

and writing $s = \rho \mathrm{e}^{\mathrm{i}\theta}, 0 \leqslant \theta \leqslant \pi$, we have

$$-\mathrm{i}s\,\mathrm{e}^{\mathrm{i}\varphi} I_{-is}(x) \sim \exp\left\{\rho \sin\theta \log(x/2\rho) + \frac{1}{2}\log\frac{\rho}{2\pi} - \rho(\theta - \tfrac{1}{2}\pi)\cos\theta + \rho\sin\theta\right\}$$

following the use of Stirling's formula for the gamma function. In this expression the argument φ is given by

$$\varphi = (\theta - \tfrac{1}{2}\pi)(\rho \sin\theta - 1/2) - \rho\cos\theta \ln(2\rho/ex).$$

It is seen that if $0 < \delta \leqslant \theta \leqslant \pi - \delta < \pi$, then $|I_{-is}(x)| = O(\exp(-\rho \ln\rho \sin\delta))$, so that on $|s| = \rho$,

$$\lim_{\rho\to\infty} \int_{\theta=\delta}^{\pi-\delta} |I_{-is}(x)\, s^m \exp(\lambda s)|\, \mathrm{d}\theta \to 0, \tag{A 2}$$

$\forall\, m, \lambda \in C$. Consider separately the domain $0 \leqslant \theta \leqslant \delta$. Under this restriction,

$$|2s I_{-is}(x)\,\mathrm{sech}(\pi s/2)| \sim \left(\frac{\rho}{2\pi}\right)^{1/2} \exp\left\{\rho\sin\theta \ln\left(\frac{ex}{2\rho}\right)\right\} \exp(-\rho\theta\cos\theta).$$

For $\pi - \delta \leqslant \theta < \pi$, a similar result is obtained by noting that if $\varpi = \pi - \theta \Rightarrow \sin\varpi =$

$\sin\theta$, and $(\theta-\frac{1}{2}\pi)\cos\theta=(\varpi-\pi/2)\cos\varpi$. It is concluded that Jordan's lemma is applicable to

$$\int_\gamma I_{-is}(x)\,s^\nu\,\mathrm{sech}(\pi s/2)\,\mathrm{d}s$$

where γ is the usual semicircular arc in the upper half-plane and $\mathrm{Re}(\nu)<1/2$. There is a need to observe, for the purposes of using the residue theorem in § 10 to obtain the near-field expansion, that the above result on the arc γ can be extended to the integral

$$\int_\gamma A_-(s)\,I_{-is}(\kappa R)\,\frac{\cosh s\alpha\coth\pi s}{\sinh\pi s}\,\mathrm{d}s.$$

To see this, note firstly that, from the result in Appendix B, $A_-(s)$ is dominated by the two terms $-2d_1\sinh s(\pi/2-\alpha+\mathrm{i}\sigma)$ and $2d_{J-1}\sinh s(\frac{1}{2}\pi-\alpha-\mathrm{i}\sigma)$ on each of the two arcs $0\leqslant\theta\leqslant\delta$ and $\pi-\delta\leqslant\theta<\pi$. Denoting the union of the arcs by $\delta\gamma$, the estimate

$$\left|\int_{\delta\gamma} A_-(s)\,I_{-is}(\kappa R)\,\frac{\cosh s\alpha\coth\pi s}{\sinh\pi s}\,\mathrm{d}s\right|$$
$$=O\left\{\frac{1}{(2\pi\rho)^{1/2}}\int_{\theta=0}^{\delta}\exp\left[\rho(\pm\sigma+\log(xe/2\rho))\,\sin\theta\,\mathrm{d}\theta\right\}\right.$$

is available, confirming that this contribution dies out like $1/(\rho^{3/2}\ln\rho)$ as $\rho\to\infty$. The behaviour on $\gamma-\delta\gamma$ is covered by (A2).

(ii) Far-field asymptotics of the solution integral (3.1) are normally obtained by expanding the integrand for large $|s|$ and inverting exactly a finite number of terms that arise. The estimate of a tail can usually be covered by the following theorem:

THEOREM. *If $f(s)\in L(0,\infty)$, and if for some s_0, $|f(s)|<\exp(-\lambda s)$ for some $\lambda>0,\forall s>s_0$ then*

$$F(x)\equiv\int_0^\infty f(s)K_{is}(x)\,\mathrm{d}s\sim\mathrm{e}^{-x}(\pi/2x)^{1/2}\int_0^\infty f(s)\mathrm{d}s.$$

Proof. Select an arbitrary positive number $N>s_0$ and denote $s_1=x-x^{1/4}$, $s_2=x+x^{1/4}$ and write

$$F(x)=\sum_{r=0}^{3}I_r=\int_0^N+\int_N^{s_1}+\int_{s_1}^{s_2}+\int_{s_2}^\infty f(s)K_{is}(x)\,\mathrm{d}s.$$

An estimate of I_0 is given by the conventional large-argument asymptotic expansion for the Macdonald function, thus

$$I_0=\mathrm{e}^{-x}(\pi/2x)^{1/2}\int_0^N f(s)\mathrm{d}s+O(\mathrm{e}^{-x}x^{-3/2}).$$

In the regime $s\leqslant s_1$ there is (Magnus, Oberhettinger & Soni pp. 139–142)

$$K_{is}(x)\sim(\tfrac{1}{2}\pi)^{1/2}(x^2-s^2)^{-1/4}\exp\left\{-(x^2-s^2)^{1/2}-s\,\sin^{-1}(s/x)\right\}\times(1+O(x^{-1})).$$

Note that $(x^2-s^2)^{1/2}+s\,\sin^{-1}(s/x)$ is an increasing function. Writing $s=x\sin\theta$ and denoting respectively $\theta_1=\sin^{-1}(s_1/x)$, $\theta_0=\sin^{-1}(N/x)$, it follows that

$$|I_1|\leqslant\left(\frac{\pi x}{2}\right)^{1/2}\int_{\theta_0}^{\theta_1}\mathrm{e}^{-x(\cos\theta+(\theta+\lambda)\sin\theta)}(\cos\theta)^{1/2}\,\mathrm{d}\theta.$$

In order adequately to estimate $|I_1|$ it is necessary further to divide this integration range so that on the left-hand interval an inequality of the type $(\cos\theta)^{1/2} < \mu(\theta+\lambda)\cos\theta$, with μ independent of x, can be exploited which facilitates an exact integration, whilst on the right-hand interval the required bound is obtained by the smallness of the exponential in the integrand. Clearly, a possible point of division is given by $\theta_2 = \sin^{-1}(x^{-1/2})$. Proceeding as suggested, with $x > K$, for some constant K, any μ subject to $\mu > (1-1/K)^{-4}/\lambda$ may be selected. With the obvious notation, the estimates obtained are

$$|I_1^{(0)}| \leqslant \mu\left(\frac{\pi}{2x}\right)^{1/2}\left[\mathrm{e}^{-x(\cos\theta+(\theta+\lambda)\sin\theta)}\right]_{\theta_2}^{\theta_0}, \quad |I_1^{(1)}| = O(x^{3/2}\mathrm{e}^{-x}\exp(-\lambda x^{1/2}))$$

from which

$$|I_1| \leqslant \frac{1}{\lambda}\left(\frac{\pi}{2x}\right)^{1/2}\mathrm{e}^{-x-\lambda N}\{1+O(x^{-1})\}.$$

To estimate I_2 use is made of the expansion in the so-called 'transitional regime'

$$K_{is}(x) \sim \frac{\mathrm{i}}{\sqrt{12}}\Gamma\left(\frac{1}{3}\right)\left(\frac{x}{6}\right)^{-1/3}\exp\left\{-\frac{\pi s}{2}-\frac{2\pi\mathrm{i}}{3}\right\}\times\{1+o(1)\}.$$

From this it is easy to find a constant A such that $|I_2| \leqslant A\mathrm{e}^{-\pi x/2}\sinh(\pi x^{1/4}/2)\times\{1+o(1)\}$. If $x > \sqrt[3]{81}$ then $\mathrm{e}^{-\pi x/2}\sinh(\pi x^{1/4}/2) < \exp(-1.04x)$, so that $|I_2| = o(\exp(-1.04x))$. Finally, in $s \geqslant s_2$

$$K_{is}(x) \sim (2\pi)^{1/2}(s^2-x^2)^{-1/4}\mathrm{e}^{-\pi s/2}\left\{\sin\left[s\cosh^{-1}(s/x)+\tfrac{1}{4}\pi-(s^2-x^2)^{1/2}\right]+O(x^{-1})\right\}$$

and there follows

$$|I_3| \leqslant (2\pi)^{1/2}\int_{s_2}^{\infty}(s^2-x^2)^{-1/4}\mathrm{e}^{-\pi s/2-\lambda s}\mathrm{d}s = o(\exp(-\pi x/2)).$$

Putting the results together in the form

$$F(x) = \mathrm{e}^{-x}\left(\frac{\pi}{2x}\right)^{1/2}\int_0^{\infty} f(s)\,\mathrm{d}s + R_N$$

it is seen that

$$|R_N| \leqslant |I_1|+|I_2|+|I_3|+\lambda^{-1}x^{-1/2}\mathrm{e}^{-x-\lambda N}\left(\tfrac{1}{2}\pi\right)^{1/2} \leqslant \left(\tfrac{1}{2}\pi x\right)^{1/2}\lambda^{-1}\mathrm{e}^{-x}\{2\mathrm{e}^{-\lambda N}+o(1)\}.$$

□

Appendix B

Consider the reduction of $A_\pm$ to hyperbolic polynomial form. From (4.5)

$$A_+ = \frac{1}{\sinh s\alpha}\sum_{j=0}^{J-1} c_j \sinh s(2j\alpha - \mathrm{i}\sigma - \tfrac{1}{2}\pi). \tag{B 1}$$

Denote, for convenience,

$$S(k) \equiv \frac{\sinh sk\alpha}{\sinh s\alpha} = \sum_{r=1}^{k}\mathrm{e}^{(k-2r+1)\alpha s},$$

and recall, from the text, that $C(s)$ has a zero at $s=0$; the implication of this is that

$$\sum_{j=0}^{J-1} c_j = 0. \tag{B 2}$$

It follows that, when (B 1) is expanded, $\sum_{j=0}^{J-1} c_j \cosh 2j\alpha s$ may be replaced by $\sum_{j=1}^{J-1} c_j \{\cosh 2j\alpha s - 1\}$, a result which enables (B 1) to be re-expressed in the form

$$A_+ = 2\sum_{j=1}^{J-1} c_j\, S(j) \cosh s(\tfrac{1}{2}\pi - j\alpha + i\sigma). \tag{B 3}$$

Replace $S(j)\cosh s(\frac{1}{2}\pi - j\alpha + \mathrm{i}\sigma)$ by exponentials, to give

$$A_+ = \sum_{j=1}^{J-1} c_j \sum_{r=1}^{j} \left\{ \mathrm{e}^{s(\frac{1}{2}\pi-(2r-1)\alpha+\mathrm{i}\sigma)} + \mathrm{e}^{-s(\frac{1}{2}\pi-2j\alpha+\mathrm{i}\sigma+(2r-1)\alpha)} \right\}$$

whereby, if in the second term of the inner summation, the summation variable is changed by $j-r=k-1$ then, after also changing the order of the repeated summation,

$$A_+ = 2\sum_{r=1}^{J-1} d_r \cosh s\left(\tfrac{1}{2}\pi - (2r-1)\alpha + \mathrm{i}\sigma\right),$$

where $d_r = \sum_{j=r}^{J-1} c_j$.

A similar treatment for A_- yields the result

$$A_- = -2\sum_{r=1}^{J-1} d_r \sinh s\left(\tfrac{1}{2}\pi - (2r-1)\alpha + \mathrm{i}\sigma\right).$$

Special case

For the case $N=0$, $J=4$, i.e. $\alpha = \pi/6$, there follows

$$\begin{aligned}\phi = \phi_\infty &+ \surd 3\,(1-4\kappa^{-2})\mathrm{e}^{-R\cos(\theta+\alpha)} \cos[R\sin(\theta+\alpha)(1-\kappa^2)^{1/2}] \\ &+ \mathrm{e}^{-R\sin(\theta+2\alpha)} \Big\{ \surd 3 \cos[R\cos(\theta+2\alpha)(1-\kappa^2)^{1/2}] \\ &\qquad + 2\sinh 2\sigma \sin[R\cos(\theta+2\alpha)(1-\kappa^2)^{1/2}] \Big\},\end{aligned}$$

where

$$\phi_\infty = \mathrm{e}^{R\sin\theta} \left\{ \surd 3 \cos[R\cos\theta\,(1-\kappa^2)^{1/2}] + 2\sinh 2\sigma \sin[R\cos\theta\,(1-\kappa^2)^{1/2}] \right\}.$$

It is easy to verify that the above potential satisfies all the conditions (2.2)–(2.6).

Appendix C

A proof is given of the lemma

$$\left(\frac{N}{2}\right)^{1/2} = N\sin\left\{\frac{(2N+1)\pi}{4}\right\} + \sum_{k=1}^{N-1} \frac{k\sin\frac{1}{2}\pi k(1+\frac{1}{2N})}{\cos(N-k)\alpha} \prod_{r=1}^{N-k} \cot r\alpha; \quad \alpha = \pi/4N,$$

which was used in § 7.

Define first $c(s) = 2^{m-1}\prod_{j=0}^{m-1} \cos\alpha(s+j)$; $m = 2N$. Then observe that writing $x = \mathrm{e}^{\mathrm{i}\theta}$ and allowing the limit $\theta \to 0$ in the identity

$$x^m - x^{-m} \equiv (x - x^{-1})\prod_{j=1}^{m-1}(x - 2\cos 2\alpha j + x^{-1})$$

leads to the result $m = 2^{2(m-1)}\prod_{j=1}^{m-1}\sin^2 \alpha j = 2^{2m-2}\prod_{j=1}^{m-1}\cos^2 \alpha j$, so that $c(0) = m^{1/2}$. Note also that $c(s)$ satisfies

$$c(s+1) = -c(s)\tan s\alpha, \tag{C 1}$$

a result which when differentiated leads to the further observation $c'(1) = -\alpha m^{1/2}$.

Now it is clear from expanding the definition of $c(s)$ that a suitable alternative form is given by

$$c(s) = \sum_{j=-m}^{m} p_j\,\mathrm{e}^{\mathrm{i}\alpha js}; \quad p_m = \mathrm{e}^{\frac{1}{4}\mathrm{i}\pi}(m-1), \quad p_{m-1} = 0, \tag{C 2}$$

and, by premultiplying the difference equation (C 1) by $\mathrm{e}^{-\mathrm{i}\pi js/2m}\cos s\alpha$ and integrating over $[0, 4m]$, the recurrence relation becomes

$$p_{j-1}\cos\frac{\alpha}{2}(j-1+m) = \mathrm{e}^{-\mathrm{i}\alpha(m-1)}\,p_{j+1}\sin\tfrac{1}{2}\alpha(j+1+m).$$

It follows that $p_{m-1} = 0 \Rightarrow p_{m-3} = 0 \Rightarrow \ldots \Rightarrow p_{1-m} = 0$. It also follows that $p_j = \bar{p}_{-j}$ (bar denoting the complex conjugate). Next differentiate (C 2) so that

$$(N/2)^{1/2} = \sum_{j=1}^{N}\mathrm{Im}(b_j\,j\,\mathrm{e}^{2\mathrm{i}\alpha j}) \tag{C 3}$$

using the alternative notation $b_{\pm(N-j)} = 2p_{\pm(m-2j)}$. For the b_j observe that

$$b_N = \mathrm{e}^{\frac{1}{4}\mathrm{i}\pi(2N-1)},$$

$$b_{N-1} = \frac{\sin\frac{1}{2}\alpha(4N)}{\cos\frac{1}{2}\alpha(4N-2)}\,\mathrm{e}^{\mathrm{i}(N-1)(\frac{1}{2}\pi-\alpha)},$$

$$b_{N-2} = \frac{\sin\frac{1}{2}\alpha(4N)}{\cos\frac{1}{2}\alpha(4N-2)}\frac{\sin\frac{1}{2}\alpha(4N-2)}{\cos\frac{1}{2}\alpha(4N-4)}\,\mathrm{e}^{\mathrm{i}(N-2)(\frac{1}{2}\pi-\alpha)},$$

$$b_{N-3} = \frac{\sin\frac{1}{2}\alpha(4N)}{\cos\frac{1}{2}\alpha(4N-2)}\frac{\sin\frac{1}{2}\alpha(4N-2)}{\cos\frac{1}{2}\alpha(4N-4)}\frac{\sin\frac{1}{2}\alpha(4N-4)}{\cos\frac{1}{2}\alpha(4N-6)}\,\mathrm{e}^{\mathrm{i}(N-3)(\frac{1}{2}\pi-\alpha)},$$

so that, in general

$$b_{N-j} = \frac{\prod_{r=1}^{j}\cot\left(\pi r/4N\right)}{\cos\left(j\pi/4N\right)}\,\mathrm{e}^{\mathrm{i}(N-j)(\frac{1}{2}\pi-\alpha)},$$

which result means that (C 3) may be expressed as

$$(N/2)^{1/2} = \sum_{j=1}^{N}\frac{j\sin\frac{1}{2}\pi j(1+\frac{1}{2N})}{\cos(\pi/4N)(N-j)}\prod_{r=1}^{N-j}\cot\left(\frac{\pi r}{4N}\right)$$

with the product taken as unity when $j = N$.

Appendix D. Proof of solution

A careful examination of the asymptotics for large X of $K_{is}(\cdot)$ on the line L, defined by $s = X + \mathrm{i}y$ for $0 \leqslant y \leqslant 1$, reveals that condition C3 (see § 3) cannot be satisfied without decomposition of $A(s)$. The Macdonald function decays only like $X^{y-1/2}\exp(-\pi X/2)$ and the failure of C3 is easily noted by taking account of (4.8). It is therefore necessary to establish the solution by a demonstration that (2.2)–(2.6) are all satisfied by the proposed expression (4.6).

The differentiation under the integral sign, which establishes (2.2) and (2.3) is justified by the asymptotics which, since $\theta < 0$, implies that all integrals, for whatever value of θ, decay exponentially (see Appendix A) and uniformly so w.r.t. θ. Details of asymptotics of the Macdonald function are given in that Appendix. The conditions (2.5) and (2.6) were dealt with in the text. It remains therefore to establish rigorously the satisfaction of (2.4). To do this, consider first

$$\phi^{(2)}(R,\theta) = \int_0^\infty A_-(s)\coth \pi s \cosh s(\theta+\alpha)\, K_{is}(\kappa R)\,\mathrm{d}s\,.$$

A partitioning of the integrand is now required such that the dominant asymptotic terms may be inverted exactly. This technique, albeit with slight variation, will also prove useful in the subsequent computations. Use is made of the fact that, near the water surface, $\cosh s(\theta+\alpha)$ may be replaced by $\sinh s(\theta+\alpha)$ in asymptotic expressions. Thus another, if evidently more convoluted, way of expressing the asymptotics of $A_-(s)\cosh s(\theta+\alpha)$ is, from (4.8),

$$\begin{aligned} &A_-(s)\cosh s(\theta+\alpha)\coth \pi s \\ &\qquad \sim \left\{c_{J-1}\cosh s\left(\tfrac{1}{2}\pi+\theta-\mathrm{i}\sigma\right) + c_0 \cosh s\left(\tfrac{1}{2}\pi+\theta+\mathrm{i}\sigma\right)\right\} \equiv A_-^\infty(s)\end{aligned}$$

from which, after some manipulation, it may further be seen that,

$$\begin{aligned} A_2 &\equiv A_-(s)\cosh s(\theta+\alpha)\coth \pi s - A_-^\infty(s) \\ &= \frac{\cosh s(\theta+\alpha)}{\sinh s\alpha}\coth \pi s\left\{\sum_{j=1}^{J-2} c_j \cosh s(2j\alpha - \tfrac{1}{2}\pi - \mathrm{i}\sigma)\right\} + \frac{1}{\sinh s\alpha \sinh \pi s} \\ &\quad \times [c_0\{\cosh s\theta \cosh s(\pi-\alpha)\cosh s(\tfrac{1}{2}\pi+\mathrm{i}\sigma) \\ &\qquad\qquad + \sinh s\theta \sinh s\alpha \cosh s(\tfrac{1}{2}\pi - \mathrm{i}\sigma)\} + c_{J-1}\{c.c.\}], \end{aligned}$$

where by c.c. is meant the complex conjugate of the terms in the previous brace. Note that the expression for A_2 is $O(\exp(\pi s/2 - \alpha s))$ uniformly on L.

Write also $\phi^{(2)}_\infty \equiv \int_0^\infty A_-^\infty(s)\,K_{is}(\kappa R)\mathrm{d}s$, then inverting this gives the two standing waves

$$\phi_\infty{}^{(2)} \equiv \tfrac{1}{2}\pi \mathrm{e}^{R\sin\theta}\left\{a_{J-1}\mathrm{e}^{-\mathrm{i}R\cos\theta(1-\kappa^2)^{1/2}} + a_{-J+1}\mathrm{e}^{\mathrm{i}R\cos\theta(1-\kappa^2)^{1/2}}\right\}$$

and it follows trivially that $L\phi_\infty{}^{(2)}|_{\theta=0} = 0$ where the operator L is defined by $L[\phi] \equiv R^{-1}\partial\phi/\partial R - \phi$. Furthermore,

$$\begin{aligned} 2L\phi^{(2)}|_{\theta=0} = &\int_{-\infty}^\infty \frac{s\,K_{is}(\kappa R)}{R}\left[A_-\sinh s\,\alpha \coth \pi s - c_{J-1}\sinh s(\tfrac{1}{2}\pi - \mathrm{i}\sigma)\right. \\ &\qquad\qquad \left. - c_0 \sinh s(\tfrac{1}{2}\pi + \mathrm{i}\sigma)\right]\,\mathrm{d}s \\ &- \int_{-\infty}^\infty K_{is}(\kappa R)\left[A_-\cosh s\alpha \coth \pi s - c_{J-1}\cosh s(\tfrac{1}{2}\pi - \mathrm{i}\sigma)\right. \\ &\qquad\qquad \left. - c_0\cosh s(\tfrac{1}{2}\pi+\mathrm{i}\sigma)\right]\,\mathrm{d}s, \end{aligned}$$

the differentiation under the integral sign now being justified by the asymptotics; details of problems similar to this are fully discussed by Ehrenmark (1989). As indicated in § 3, it is now possible to replace $sK_{is}(\kappa R)/R$ in the first integral by an expression involving K_{is-1} and K_{is+1}, and make substitution $s = s' \pm \mathrm{i}$, so that the Macdonald function recovers the order is in all expressions. The result will be two new integrals along $\mathrm{Im}(s') = \pm\mathrm{i}$, but these can now be deformed to the real axis (indented respectively above and below the origin) by Cauchy's theorem, since the integrand now decays like $\exp(-2s'\alpha)$ on both L and its mirror image in the imaginary axis. Write, for convenience, $a^{\pm} \equiv a_{J-1} \pm a_{-J+1}$. The result of the above is then

$$
\begin{aligned}
&\frac{4}{\mathrm{i}\kappa} L\phi^{(1)}\,|_{\theta=0} \\
&\quad = \int_{-\infty}^{\infty} K_{is}(\kappa R) \Big\{ A_-(s-\mathrm{i}) \sinh(s-\mathrm{i})\alpha - A_-(s+\mathrm{i}) \sinh(s+\mathrm{i})\alpha \\
&\quad -\frac{2}{\mathrm{i}\kappa} A_-(s) \cosh s\alpha \Big\} \coth \pi s \, \mathrm{d}s \\
&\quad - \int_{-\infty}^{\infty} K_{is}(\kappa R) \Big\{ c_{J-1} [\sinh(s-\mathrm{i}) \left(\tfrac{1}{2}\pi - \mathrm{i}\sigma\right) - \sinh(s+\mathrm{i}) \left(\tfrac{1}{2}\pi - \mathrm{i}\sigma\right) \\
&\quad -\frac{2}{\mathrm{i}\kappa} \cosh s \left(\tfrac{1}{2}\pi - \mathrm{i}\sigma\right)] - c_0[\mathrm{c.c.}] \Big\} \mathrm{d}s + 2 \sin \alpha \, A_-(i) K_0(\kappa R),
\end{aligned}
$$

the last term arising from the residues at the indents. Here [*c.c.*] denotes the complex conjugate of the preceding term in similar brackets. The first of the integrals vanishes since A_- satisfies construction (3.5). Observing that $\cosh \sigma = \kappa^{-1}$, it is easily verified that the second integral vanishes also. Thus $L\phi^{(1)}|_{\theta=0} = \tfrac{1}{2}\mathrm{i}\kappa \sin \alpha \, A_-(i) K_0(\kappa R)$. It follows that

$$
\phi = \lambda K_0(\kappa R) + \int_0^{\infty} A_-(s) \coth \pi s \cosh s(\theta + \alpha) K_{is}(\kappa R) \, \mathrm{d}s
$$

is a valid solution of the problem by making the choice

$$
\lambda = \tfrac{1}{2}\mathrm{i}\kappa \sin \alpha \, A_-(i).
$$

The remaining ('regular') part of the solution integral,

$$
\phi^{(1)}(R, \theta) = \int_0^{\infty} A_+(s) \cosh s(\theta + \alpha) K_{is}(\kappa R) \, \mathrm{d}s,
$$

may be dealt with in an identical fashion, except that there is no need for the indentation at the origin. The rest of the development is exactly as above and is not repeated here, in the interests of brevity. Moreover, as is well known and confirmed in the text, a closed form is available from which the solution is also readily established.

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